2. Pontriagin, L. S. and Mishchenko, E. F., The contact evasion problem in linear differential games. Differents. Uravnen., Vol. 7, No. 3, 1971.
3. Pontriagin, L. S., A linear differential game of escape. Tr. Matem. Inst., Vol. 112, 1971.
4. Krasovskii, N. N., An encounter-evasion differential game. I. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, No. 2, 1973.
5. Krasovskii, N. N., An encounter-evasion differential game. II. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, No. 3, 1973.
6. Pshenichnyi, B. N. and Chikrii, A. A., The contact evasion problem in differential games. Zh. Vychisl. Mat. mat. Fiz., (English translation), Pergamon Press, Vol. 14, No. 6, 1974.
7. Nikol'skii, M. S., On one method of escape. Dokl. Akad. Nauk SSSR, Vol. 214, No. 2, 1974.
8. Grigorenko, N. L., The contact evasion problem in nonstationary linear differential games. Kibernetika, No. 6, 1973.
9. Chikrii, A. A., The contact evasion problem for a terminal set possessing a piecewise-smooth boundary. Kibernetika, No. 5, 1974.
10. Pschenitchny, B. N., ε-strategies in differential games. Topics in differential games, edited by Austin Blaguiere, New York—London—Amsterdam, North-Holland Publ. Co., 1973.
11. Sansone, G., Ordinary Differential Equations, Vol. 1. Moscow, Izd. Inostr. Lit., 1953.
12. Kartan, A., Differential Calulus — Differential Forms. Moscow, "Mir", 1971.

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APPLICATION OF THE PERTURBATION METHOD TO SOME OPTIMAL CONTROL PROBLEMS

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We examine a quasilinear optimal control system. We justify the applicability of the perturbation method to some control problems. Various systems for constructing an approximate solution of control problems with a small parameter were presented in /1 – 4/. A number of practical optimal control problems can be described by systems containing linear terms and small, in general, nonlinear perturbing factors. The scheme of successive approximations of the perturbation method developed below, can prove to be useful for the analytic investigations of such problems. The method is justified for quasilinear systems with a quadratic performance index.

1. Let a control system be given by the equation

$$x^{\cdot}(t) = A(t)\,x(t) + \varepsilon f(x(t),\ t) + B(t)\,u(t), \quad x(0) = x_0, \quad 0 \leqslant t \leqslant T \tag{1.1}$$

where the vector $x(t)$ belongs to an n-dimensional Euclidean space E_n. The following constraints on the coefficients of system (1.1) are assumed to be constantly fulfilled. The given matrices $A(t)$, $B(t)$, $0 \leqslant t \leqslant T$ are measurable and bounded, and $\varepsilon > 0$ is some constant. The vector-valued function $f(t,\ x) \in E_n$ is measurable in both arguments and satisfies the following conditions: (1.2)

$$|f(x_1,\ t) - f(x_2,\ t)| \leqslant \alpha_1(N)\,|x_1 - x_2|, \qquad |f(x,\ t)| \leqslant \alpha_3 + \alpha_2 |x|^2$$

for any $x_1,\ x_2 \in E_n$, $|x_i| \leqslant N$ and any $0 \leqslant t \leqslant T$. Here the constants $\alpha_i \geqslant 0$ and the symbol $|x|$ denotes the Euclidean norm of vector $x \in E_n$; the constant α_1 depends, in general, on the dimensions of the regions in E_n to which $x_1,\ x_2$ belong and $\alpha_1(N)$ does not decrease. Thus, the first of requirements (1.2) signifies that the function $f(x,\ t)$ satisfies the local Lipschitz condition with respect to its first argument.

Problem 1. Choose the control $u(t) \in E_l$ which minimizes the quadratic functional (the performance index)

$$I(x, u) = x'(T)\,Hx(T) + \int_0^T [x'(t)\,L_1(t)\,x(t) + u'(t)\,L_2(t)\,u(t)]\,dt \tag{1.3}$$

Here the prime is the sign of transposition, the matrices L_1, L_2 are measurable and bounded, H and L_1 are nonnegative definite, and $L_2(t)$ is uniformly positive definite on the interval $[0,\ T]$. When $\varepsilon = 0$ the problem of minimizing functional (1.3) on the trajectories of system (1.1) admits of an explicit analytic solution (see /5/, for example). In the present paper we propose and substantiate an algorithm for constructing controls ensuring accuracy, in the sense of functional I, of order ε^2 for arbitraty sufficiently small $\varepsilon \geqslant 0$ and we find the upper bound of the values of ε for which the proposed algorithm is valid. Sections 2 and 3 are devoted to the proof of the algorithm, while its constructive part is set forth in Sect. 4.

2. Let us consider the controlled system (1.1) for $\varepsilon = 0$ with performance index (1.3). The optimal control, when $\varepsilon = 0$, is denoted by u_0, while the corresponding trajectory, by x_0. Since /5/

$$u_0(t) = -L_2^{-1}(t)\,B'(t)\,P(t)\,x(t) \tag{2.1}$$

(here L_2^{-1} is the matrix inverse to L_2), the value of functional (1.3), corresponding to control (2.1), equals

$$x'(0)\,P(0)\,x(0) \tag{2.2}$$

We note that the matrix $P(t)$ is defined only by the coefficients of the original problem (1.1),(1.3). Therefore, it can be found right at the beginning of the control process. By treating the u_0, x_0 constructed in this manner as the zero approximation, we form a sequence of controls u_k as the sequence minimizing for each $k = 1,\ 2,\ \ldots$ functional (1.3) on the trajectories of the linear system

$$x^{\cdot}(t) = A(t)\,x(t) + \varepsilon f(x_{k-1}(t),\ t) + B(t)\,u(t) \tag{2.3}$$

By x_k we denote the trajectory of system (2.3) corresponding to u_k, and by I_k, the value of functional (1.3). Here, a standard application of the dynamic programing method (see /5/, for example) shows that for $k \geqslant 1$, u_k and I_k are given by the formulas

$$u_k(t) = -L_2^{-1}(t)B'(t)\,[P(t)\,x(t) + G_{1k}'(t)] \tag{2.4}$$
$$I_k = x'(0)P(0)\,x(0) + G_{1k}(0)\,x(0) + x'(0)\,G_{2k}(0) + Q_k(0)$$

where $G_{1k},\ G_{2k},\ Q_k$ satisfy almost everywhere the equations

$$G_{1k}^{\cdot}(t) + \varepsilon f'(x_{k-1}(t),\ t)\,P(t) + G_{1k}(t)\,B_1(t)\,P(t) = 0, \tag{2.5}$$
$$G_{1k}(T) = 0$$
$$Q_k^{\cdot}(t) + G_{1k}(t)\,\varepsilon f(x_{k-1}(t),\ t) + \varepsilon f'(x_{k-1}(t), t)\,G_{1k}'(t) +$$
$$G_{1k}(t)B_1(t)\,G_{1k}'(t) = 0,\quad Q_k(T) = 0,\quad G_{1k}'(t) = G_{2k}(t)$$

Let us establish the compactness of the sequence $x_k,\ G_{1k}$ for sufficiently small ε, by using the method of /6/. From (2.5) it follows that

$$G_{1k}(t) = \varepsilon\int_t^T f'(x_{k-1}(s), s)\,P(s)\,z_1(t, s)\,ds \tag{2.6}$$

$$\frac{\partial z_1(t,s)}{\partial t} = A_1(t)\,z_1(t,s),\quad z_1(t,t) = I,\quad A_1(\tau) = A(\tau) - B_1(\tau)\,P(\tau) \tag{2.7}$$

Analogously, by virtue of (2.3),(2.4), for $k = 1, 2, \ldots$ we have

$$x_k(t) = z_1(t,0)\,x(0) + \varepsilon\int_0^t z_1(t,s)\,f(x_{k-1}(s), s)\,ds - \tag{2.8}$$
$$\varepsilon\int_0^t z_1(t,s)\,B_1(s)\,ds\int_s^T z_1(s,s_1)\,P(s_1)\,f(x_{k-1}(s_1), s_1)\,ds_1$$

We note that from the requirements imposed on the coefficients of (1.1) and (1.3) in Sect. 1 follows the existence of the bounded nonnegative-definite matrix $P(t),\ 0 \leqslant t \leqslant T$ (see /5, 7/, for example). Hence from (2.8) follows the existence of a constant m_0 such that

$$|\,x_0(t)\,| \leqslant \sup_t |z_1(t,\ 0)\,x(0)\,| = m_0 \tag{2.9}$$

for all $0 \leqslant t \leqslant T$.

Further, let us define and fix a certain number β satisfying the inequality (see (1.2))

$$\beta > (\alpha_3 + \alpha_0 m_0^2)\sup_t\int_0^T\Big[\ z_1, (t,s)\,\|\,(1 + \|B_1(s)\| \times \tag{2.10}$$
$$\int_s^T \|z_1(s,s_1)\,P(s_1)\|\,ds_1)\Big]\,ds = (\alpha_3 + \alpha_0 m_0^2)\,\beta_1 = \beta_2(m_0),\quad 0 \leqslant t \leqslant T$$

From this definition of constant β and from (2.8) it follows that

$$|\,x_1(t)\,| \leqslant m_0 + \varepsilon\beta \tag{2.11}$$

But all the $x_k(t)$ satisfy a bound similar to (2.11) for any sufficiently small ε. In fact, on the basis of (2.8), (1.2) the inequality

$$|\,x_k(t)\,| \leqslant m_0 + \varepsilon\beta \tag{2.12}$$

is valid for all $\varepsilon \geqslant 0$ for which

$$\beta_2(m_0) + \varepsilon\,(2\alpha_2 m_0\beta + \alpha_2\varepsilon\beta^2)\,\beta_1 < \beta \tag{2.13}$$

The uniform boundedness of $G_{1k}(t)$ also follows from (2.12), (2.6), (1.2).

To prove the equicontinuity of sequence $x_k(t)$ we take two arbitrary points $t_1 \leqslant t_2$ on interval $[0, T]$ and we integrate both sides of (2.3) from t_1 and t_2. Hence, with due regard to (2.4), (2.5), (2.8), (2.12). follows the equicontinuity of $x_k(t)$, and with it, that of $G_{1k}(t)$.

We set

$$\gamma_k = \max_t |x_{k+1}(t) - x_k(t)|, \qquad 0 \leqslant t \leqslant T$$

Then from (2.8), (1.2) we conclude that

$$\gamma_{k+1} \leqslant \varepsilon \gamma_k \alpha_1 (m_0 + \varepsilon\beta)\beta_1 \tag{2.14}$$

where the constant β_1 has been defined in (2.10). Let us now require that

$$\varepsilon\alpha_1 (m_0 + \varepsilon\beta)\beta_1 < 1 \tag{2.15}$$

Relations (2.14) and (2.15) permit us to estimate $|x(t) - x_k(t)|$. We have (binging in (2.8))

$$|x(t) - x_k(t)| = |\sum_{i=k}^{\infty} (x_{i+1}(t) - x_i(t))| \leqslant \tag{2.16}$$
$$\varepsilon\beta_2 (\varepsilon\alpha_1 (m_0 + \varepsilon\beta)\beta_1)^k [1 - \varepsilon\alpha_1 (m_0 + \varepsilon\beta)\beta_1]^{-1}$$

Analogously, with due regard to (2.6), we obtain

$$|G_1(t) - G_{11}(t)| \leqslant \varepsilon^2\alpha_1 (m_0 + \varepsilon\beta)\beta_2 [1 - \varepsilon\alpha_1 (m_0 + \varepsilon\beta)\beta_1]^{-1} \times \tag{2.17}$$
$$\sup_t \int_t^T P(s) z_1(t, s)\, ds$$

Finally, we note that from the uniform convergence of sequence x_k, G_{1k} to x, G_1 it follows that the limit functions x, G_1 are the solution of the boundary-value problem formed by system (1.1) and the relations

$$u(t) = -L_2^{-1}(t) B'(t) [P(t) x(t) + G_1(t)] \tag{2.18}$$
$$G_1^{\cdot}(t) + \varepsilon f'(x(t), t) P(t) + G_1(t) A(t) -$$
$$G_1(t) B_1(t) P(t) = 0, \quad G_1(T) = 0$$

Here, on the basis of (2.4), the value of functional (1.3) on the trajectory of system (1.1), corresponding to control (2.18), is

$$I_0 = x'(0) P(0) x(0) + G_1(0) x(0) + x'(0) G_2(0) + Q(0) \tag{2.19}$$
$$G_1'(t) = G_2(t)$$
$$Q^{\cdot}(t) + G_1(t) \varepsilon f(x(t), t) + \varepsilon f'(x(t), t) G_1'(t) +$$
$$G_1(t) B_1(t) G_1'(t) = 0, \qquad Q(T) = 0$$

Below we have shown that the approximations u_1, x_1 ensure an accuracy with respect to the functional of order ε^2, while approximations u_0, x_0, an accuracy of order ε.

3. 3.1. Let us prove the existence of a solution to Problem 1. We set

$$I_0 = \inf_u I \tag{3.1}$$

where the infimum is computed over the set of all admissible controls. With due regard to (2.19), $I_0 \leqslant \bar{I}_0 < \infty$. Hence there exists a sequence of controls $u_i\ (t),\ i = 1, 2, \ldots$, such that the trajectories $x_i\ (t)$ of system (1.1) and the values $I_i \leqslant \bar{I}_0$ of functional (1.3), corresponding to them, satisfy the relation

$$\lim_{i\to\infty} I_i = I_0, \quad \int_0^T u_i'(t)\, L_2(t)\, u_i(t)\, dt \leqslant \overline{I_0} \tag{3.2}$$

From (3.2) and the uniform positive definiteness of $L_2\ (t)$ follows the existence of a constant $c_1 > 0$ such that

$$\int_0^T |\, u_i(t)\,|^2\, dt \leqslant c_1 \tag{3.3}$$

uniformly with respect to i.

Let us now show that the uniform boundedness of sequence $x_i\ (t),\ 0 \leqslant t \leqslant T,\ i = 1, 2, \ldots$ follows from (3.3). For this we set $v_i = x_i' x_i$. By virtue of (1.1) and (1.2) we have $v_i\ (t) \leqslant \omega_i\ (t)$, where $\omega_i\ (t)$ is given by the Riccati equation

$$\omega_i^{\cdot}\ (t) = \omega_i\ (t)\ A_1 + \|\, B\ (t)\, \|\ |\ u_i\ (t)\ |^2 + \varepsilon \alpha_2 \omega_i^2\ (t),$$
$$\omega_i\ (0) = v_i\ (0)$$
$$A_1 = \sup_t\ (2\, \|\, A\ (t)\, \| + \|\, B\ (t)\, \| + \varepsilon + 2\varepsilon\alpha_2\alpha_3)$$

We represent $\omega_i\ (t)$ in the form $\omega_i = \omega_{i1} + \omega_{i2}$, where

$$\omega_{i2}^{\cdot}\ (t) = \omega_{i2}\ (t)\ A_1 + \|\, B\ (t)\, \|\ |\ u_i\ (t)\ |^2, \quad \omega_{i2}\ (0) = \omega_i\ (0)$$

On the basis of (3.3) all the $\omega_{i2}\ (t)$ are uniformly bounded. Hence follows the uniform boundedness of all $v_i\ (t)$, if only

$$\varepsilon < \pi/(2T \sqrt{\varepsilon_1\varepsilon_2}) \tag{3.4}$$

$$\varepsilon_1 = \alpha_2 \sup_{i,t} \omega_{i2}^2\,(t), \quad 0 \leqslant t \leqslant T$$

$$\varepsilon_2 = \alpha_2 \sup_{i,t} \omega_{i2}^2\,(t) \exp \int_0^t (A_1 + 2\varepsilon\alpha_2\omega_{i2}\,(s))\, ds$$

Thus, we have constructed, by the method indicated, a constant m_1 for which $|\ x_i\ (t)\ | \leqslant m_1$, i.e. the sequence $x_k\ (t)$ is equicontinuous. Now, the validity of the assertion in Sect. 1 can be established in standard fashion, similarly to /8 – 10/.

3.2. Let us show that from the fact of existence of the optimal control $u_0\ (t)$ follows the existence of a control $u\ (t)$ and a trajectory $x\ (t)$ of system (1.1), corresponding to $u\ (t)$, satisfying relations (1.1), (2.18), (2.19), and being such that $I\ (u,\ x)$ differs from I_0 by a quantity of order ε^2. Thus, in particular, we shall have established that the existence of a solution of the boundary-value problem (1.1), (2.18) follows from the existence of the optimal control $u_0\ (t)$. To prove this we consider an auxiliary problem of minimizing functional (1.3) on the trajectories of the linear system

$$x^{\cdot}\ (t) = A\ (t)\ x\ (t) + \varepsilon f\ (x_0\ (t),\ t) + B\ (t)\ u\ (t) \tag{3.5}$$

We denote the minimal value of functional (1.3) on the trajectories of system (3.5) by I_1. From the definition of $u_0\ (t)$ and (2.11) we conclude that $I_1 \leqslant I_0 \leqslant \bar{I}_0$. In addition, from the linearity of (2.11) and (1.3) follows the existence of the optimal control $u_1(t)$ and of the trajectory $x_1(t)$ on which the value I_1 of functional (1.3) is achieved /5/. In analogous manner we construct the optimal control $u_k(t)$ which supplies the minimum of functional (1.3) on the trajectories of the linear system (2.3). By I_k and $x_k(t)$ we denote the value of functional (1.3) and the trajectory of system (2.3) corresponding to $u_k(t)$. The subsequent construction is analogous to that in Sect. 2.

Namely, we fix a certain number $\bar\beta \geqslant \beta$ satisfying the inequality $\bar\beta > \beta_2(m_2)$, $m_2 = \max(m_0, m_1)$. Then, similar to (2.12), we get that for all ε for which

$$\begin{aligned} &\beta_2(m_0) + \varepsilon(2\alpha_2 m_0\bar\beta + \alpha_2\varepsilon\bar\beta^2)\beta_1 < \bar\beta \\ &\varepsilon\alpha_1(m_0 + \varepsilon\bar\beta)\bar\beta_1 < 1 \end{aligned} \tag{3.6}$$

the sequence $x_k(t)$, $k \geqslant 1$ is bounded. Precisely,

$$|x_k(t)| \leqslant m_0 + \varepsilon\bar\beta$$

Further, by a verbatim repetition of the arguments in Sect. 2, we get that x_k, u_k converge to x, u which satisfy (1.1),(2.18), (2.19). To estimate the difference $-I_0 + I(x, u)$ it is sufficient to estimate $I(x, u) - I_1$ since $I_1 \leqslant I_0 \leqslant I(x, u)$. However, from (2.4)–(2.8) and (2.18), using bounds of type (2.16) and (2.17), we get that

$$I(x, u) = I_1 + O(\varepsilon^2)$$

By the same token we have proved the assertion of Sect. 3.2.

3.3. Let us show that the solution of the boundary-value problem (1.1), (2.18) is unique in the region $|x| \leqslant m_0 + \varepsilon\bar\beta$. Assume the contrary. Let x_1, x_2 be two solutions of (1.1), (2.18). Then, with due regard to (2.8), we have

$$x_1(t) - x_2(t) = \varepsilon\int_0^t z_1(t,s)[f(x_1(s),s) - f_2(x_2(s),s)]\,ds - \tag{3.7}$$

$$\varepsilon\int_0^t z_1(t,s)B_1(s)\int_s^T z_1'(s,\tau)P(\tau)[f(x_1(\tau),\tau) - f(x_2(\tau),\tau)]\,d\tau ds$$

We now set

$$\beta_0 = \sup_t|x_1(t) - x_2(t)|, \quad 0 \leqslant t \leqslant T \tag{3.8}$$

Then from (3.8), (3.7) we obtain

$$\beta_0 \leqslant \beta_0\varepsilon\beta_1\alpha_1(m_0 + \varepsilon\bar\beta)$$

Hence from (3.5) we conclude that $x_1(t) \equiv x_2(t)$. From this identity follows the uniqueness of the solution of boundary-value problem (1.1), (2.18) with respect to $G_1(t)$ as well.

4. Above we have established the existence of a solution of Problem 1 and proposed an approximate method for solving it. This method consists of two stages.

1°. Using the solution of Problem 1 for a linear system (1.1) with $\varepsilon = 0$ (i.e. using (2.2)), we determine ε_0 such that relations (3.4), (3.5) are valid for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$. We emphasize that the value ε_0 depends only on the matrix $P(t)$ and on the coefficients of the controlled ssystem (1.1).

2°. For these values of ε we determine from formulas (2.1) and (2.8) the zero approximation u_0, x_0 to the optimal control and trajectory, while from formulas (2.4), (2.6), (2.8), the first approximation u_1, x_1.

We note that by virtue of (2.1) and (2.4) the zero and first approximations constructed to the optimal control are obtained in a form suitable for synthesis (i.e. as functions of the phase coordinates).

Let us consider certain estimates of the closeness of the optimal values of functional (1.3) for the original controlled system (1.1) and for the auxiliary dynamic system (2.3). We start with the zero approximation which is determined by formulas (2.1), (2.8) with $\varepsilon = 0$, while the corresponding value of functional (1.3) equals

$$x'(0)\,P(0)\,x(0) \tag{4.1}$$

From (2.18), (2.6) we conclude that

$$|G_1(t)| \leqslant \varepsilon\,(\alpha_3 + \alpha_2\,(m_0 + \varepsilon\beta)^2)\,\sup_t \int_t^T \|P(s)\,z_1(t,s)\|\,ds \tag{4.2}$$

Relations (4.1), (4.2), (2.19) show that the difference between the optimal value of functional (1.3) and the zero approximation to it is of order ε.

The first approximation is given by formulas (2.4) – (2.8) in which we should set $k = 1$. Using (2.4) – (2.8), (2.16), (2.17) with $k = 1$, analogously to the zero approximation, we get that the difference between the optimal value of functional (1.3) and the first approximation to it is of the order ε^2 since the functions Q and Q_k are the squares of the quantities already obtained.

We now turn to a more interesting question which is the following. We take the control $u_k, k = 0, 1$, determined by (2.4), and we control system (1.1) using u_k. We are required to determine the allowable error in functional (1.3) when the optimal control in system (1.1) is replaced by u_k Here it is sufficient to establish a bound of the form (2.16), since from it follows, completely analogously to (4.2) and (2.17), a bound on the accuracy with respect to the functional. Thus, we take some arbitrary fixed k and we estimate the difference between the solution of problem (1.1), (2.18), (2.19) and the function $y(t)$ defined by the following relations:

$$y^{\cdot}(t) = A(t)\,y(t) + \varepsilon f(y(t),\ t) - B_1(t)\,(P(t)\,y(t) + G_{1k}{}'(t)) \tag{4.3}$$
$$y(0) = x(0), \quad 0 \leqslant t \leqslant T$$

where $G_{1k}(t)$ is a solution of Eq. (2.5). It can be shown that the solution $y(t)$ of the Cauchy problem (4.3) exists and satisfies bound (2.12).

We now set $r(t) = x(t) - y(t)$. Then on the basis of (1.1), (2.18), (4.3), (2.7)

$$|r(t)| \leqslant \int_0^t [(A_1(s) + \varepsilon\alpha_1(m_0 + \varepsilon\beta))\,|r(s)| + 8\,|G_{1k}(s) - G_1(s)|]\,ds$$

Hence, applying the Gronwall-Bellman lemma, in view of (2.17) we obtain the desired bound on the difference $x(t) - y(t)$. Here the estimate of the accuracy with respect to the functional can now be established similarly to (2.17) and (4.2). This estimate shows that the control in the zero approximation (2.1) ensures an accuracy with respect to the functional of order ε, while the control in the first approximation (2.4), is of the order ε^2.

Example. As an illustration of the method developed we consider a planar problem of the optimal descent of an axisymmetric controlled object with due regard to the decelerating force of the atmosphere. Assuming that the motion is stabilized, we write the system of equations of the center of mass in the form /11/

$$v^{\cdot} = \frac{u}{m} - \frac{c_x(M)}{m} S\rho(y)\frac{v^2}{2} - g(R)\sin\theta$$

$$\theta^{\cdot} = \left[\frac{v^2}{R} - g(R)\right]\frac{\cos\theta}{v}, \quad R^{\cdot} = v\sin\theta$$

Here v is the absolute value of the velocity, θ is the path angle, R is the distance from the Earth's center to the craft's center of mass, $m \approx \text{const}$ is the mass, u is the thrust, $c_x(M)$ is the dimensionless drag coefficient, M is the Mach number, S is the area of the midsection, $\rho(y)$ is the atmospheric density, $g(R)$ is the freefall acceleration, $y = R - R_0$, R_0 is the Earth's radius. It is assumed that the balanced angle of attack can be neglected since we take $\theta \approx -{}^1/_2\pi$ in what follows.

If we assume that gravitational force acting on the body is negligibly small in comparison with the remaining forces, i.e. thrust and drag, and if we take it that the motion is close to a vertical fall, i.e. $\theta = -\pi/2 + \Delta\theta$, where the quantity $\Delta\theta^2$ can be neglected, then the system of equations simplifies considerably and by virtue of (4.4) takes the form

$$dT/dR = -u + \gamma(R, T)\rho(y)T, \qquad 2T = mv^2 \tag{4.5}$$
$$d\cos\theta / dR = [g(R) - v^2/R]$$

where T is the object's kinetic energy, $\gamma = m^{-1}c_xS$ is the ballistic coefficient. We should note that with the indicated accuracy $\sim \Delta\theta^2$ the first equation can be integrated independently of the second one. The atmospheric density (see /11/) can be taken equal to

$$\rho(y) = \rho_0 \exp(-\delta_1 y) \tag{4.6}$$

where ρ_0, δ_1 are given constants. We assume further that the ballistic coefficient γ is

$$\gamma = c_0 + \varepsilon\delta_2\left(\frac{2T}{mc^2}\right)^2 \exp\left[-\delta_3\left(\frac{2T}{mc^2} - 1\right)^2\right] \tag{4.7}$$

where c_0, δ_2, δ_3 are given constants, c is the speed of sound in the medium. This assumption can be taken as fulfilled for certain types of wingless crafts /11/. We shall measure the altitude y from the point of intersection of the vertical along which the body is descending to the Earth's surface and we shall suppose that the numbers y_1, y_2, $y_1 \geqslant y_2$ are given. The value of the kinetic energy T at altitude y_1 is given and equals $T(y_1)$.

Problem 2. Choose the control u in (4.5) so as to minimize the functional

$$I = T^2(y_2) + L_2\int_{y_2}^{y_1} u^2(s)\,ds \tag{4.8}$$

We note that by varying the parameter L_2 in (4.8) we can regulate the value of the kinetic energy at altitude y_2. Let us first consider the auxiliary problem of minimizing functional (4.8) on the trajectories of (4.5) without constraining the direction of the velocity vector, and next let us show that the solutions of this auxiliary problem for small ε yield as well the solution of Problem 2. On the basis of (4.5) – (4.8) the auxiliary problem posed reduces to Problem 1. In addition, the theorem's requirements are fulfilled on the basis of (4.5) – (4.7), which in the case being examined yields the formulas for the successive approximations to the optimal value of functional (4.8).

We cite certain calculations as an illustration. On the basis of (4.5) – (4.7) the optimal control in the zero approximation, solving the auxiliary problem, is

$$u(y) = L_2^{-1}[T(y)P(y_1 - y)] \tag{4.9}$$

Here, in view of (2.1), (2.2)

$$P(h) = \left[z(h, y_1 - y_2) + \int_h^{y_1 - y_2} B_1 z(h, s)\,ds\right]^{-1}, \quad B_1 = L_2^{-1}$$

$$\frac{\partial z(h, s)}{\partial h} = -2c_0\rho(y_1 - h)\,z(h, s), \quad z(s, s) = 1$$

From Eq. (4.5) with $\varepsilon = 0$ and from (4.6) – (4.9) it is easy to determine the zero approximation $T_0(y)$, and next, using (2.4) and (2.5), the first approximation T_1, u_1 of the solution of the auxiliary problem.

Now, to solve Problem 2 it suffices to note that the approximations T_{1k}, u_k, $k = 0, 1, \ldots$, constructed in the auxiliary problem yield, for small ε, the approximations for Problem 2 as well. In fact, for this it is sufficient only to verify that the successive approximations T_{1k} do not vanish. For T_0 this follows in view of (4.9) and (4.5) from the homogeneity of the equation satisfied by T_0. Using this fact, the uniform boundedness (see Sect. 2) of the successive approximations of the auxiliary problem, we get that the approximation T_1 does not vanish for an appropriate choice of ε.

Note. We note that the descent problem with due regard to the force of gravity can be analyzed in similar manner. Here the descent problem can be reduced to Problem 1 by imposing constraints derived from the requirement that the successive approximations T_k, $k = 0,1$ in the auxiliary problem do not vanish.

REFERENCES

1. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1967.
2. Chernous'ko, F. L., Some problems of optimal control with a small parameter. PMM Vol.32, No.1, 1968.
3. Moiseev, N. N., Asymptotic Methods in Nonlinear Mechanics. Moscow, "Nauka", 1969.
4. Al'brekht, E. G., On the control of motion in nonlinear systems. Differents. Uravnen., Vol.2, No.3, 1966.
5. Krasovskii, N. N., Problems of stabilization of controlled motions. Supplement to: Malkin, I. G., Theory of Stability of Motion. Moscow, "Fizmatgiz", 1966.
6. Malkin, I.G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
7. Roitenberg, Ia. N., Automatic Control. Moscow, "Nauka", 1971.
8. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., Mathematical Theory of Optimal Processes. Moscow, Fizmatgiz, 1969. (See also English translation L. S. Pontriagin, The Mathematical Theory of Optimal Processes, Pergamon Press, Book No. 10176, 1964).
9. Gabasov, R. and Kirillova, F. M., Qualitative Theory of Optimal Processes. Moscow, "Nauka", 1970.
10. Lee, E. B. and Markus, L., Foundations of Optimal Control Theory. Moscow, "Nauka", 1972.
11. Gorbatenko, S. A., Makashov, E. M., Polushkin, Iu. F. and Sheftel', L. V., Mechanics of Flight. Moscow, Mashinostroenie, 1969.

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